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An Edge-Corrected Linearization Technique for Boundary-Layer Problems

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FOR parabolic marching problems, as exemplified by boundary-layer calculations, implicit finite-difference procedures are preferable to explicit ones due to the lack of a stability restriction on the size of the marching step. The penalty paid for this theoretically unlimited step size is the need to solve a nonlinear set of equations at each step. This can be overcome by an efficient iteration scheme,[1] or any number of linearization techniques; see for example, the methods used in Refs. 2-4. Probably the most common linearization used is a simple quadratic extrapolation from previously calculated points, which is accomplished by a direct Taylor expansion about the desired point. This Note will show that this extrapolation procedure introduces errors into the computed solution if the imposed boundary conditions vary with the marching variable. A simple and effective alteration to the usual procedure, called edge-corrected linearization (ECL), is introduced to alleviate the difficulty.

If, during a marching procedure, we wish to advance from a point (i) to a point $(i+1)$, and center the differencing about the point $(i+½)$, i.e., a Crank-Nicolson type second-order-accurate integration, the extrapolated nonlinear coefficients (denoted by capitals) may be expressed for a given j as

$$U=u_{i+½}=\frac{3}{2}u_i-\frac{1}{2}u_{i-1}+\frac{3}{2}(\Delta x)^2u''_{i+½} \tag{1a}$$

if the function values are stored at points (i), $(i-1)$, etc., and

$$W=w_{i+½}=2w_{i-½}-w_{i-3/2}+4(\Delta x)^2w''_{i+½} \tag{1b}$$

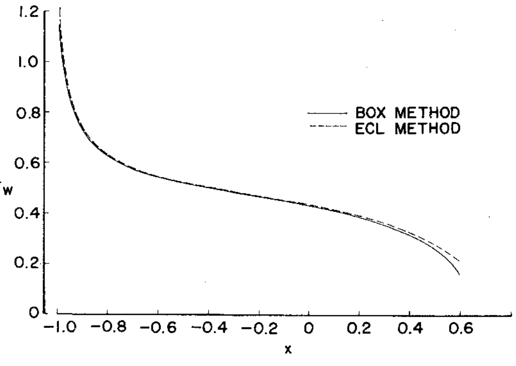

Fig. 1 Comparison of skin-friction calculations for a prolate spheroid at 0° incidence.

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if the values are stored at the midpoints, where Δx is the step size. An example of the use of this formulation in a standard boundary-layer calculation is given below.

The boundary-layer equations for incompressible flow past an arbitrary body of revolution are

$$(1/h)uu_x+wu_z=-(1/h)p_x+u_{zz} \tag{2}$$

$$(1/h)u_x+K_{21}u+w_z=0 \tag{3}$$

where the Reynolds number has been scaled into z and w. To integrate these equations, Eq. (2) is cast in difference form centered at a mid-step point, $(i+½)$, with any nonlinear values evaluated by relations (1). That is

$$\frac{1}{h_{i+½}}U\left(\frac{u_{i+1,j}-u_{i,j}}{\Delta x}\right)+W\left[\frac{1}{2}(\delta_z u_{i,j}+\delta_z u_{i+1,j})\right]$$
$$=-\frac{1}{h_{i+½}}p_{x_{i+½}}+\left[\frac{1}{2}(\delta_z^2u_{i,j}+\delta_z^2u_{i+1,j})\right] \tag{4}$$

where δ_z and δ_z^2 are the standard centered difference operators. The differencing represented by Eq. (4) yields a tridiagonal system for the unknowns $u_{i+1,j}$ which can be easily solved. Once this is done, the continuity equation, (3), can be written for w at the midpoint $(i+½)$ and integrated out from the wall to get the new values of $w_{i+½,j}$.

This fairly standard approach to the solution of simple boundary-layer problems can lead to spurious effects when the values of the velocity in Eqs. (2) and (3) are not normalized by their edge values (and hence equal to unity at each marching step), but allowed to vary with x. These effects can be traced directly to the extrapolation scheme, Eqs. (1). Consider Eq. (2) at the edge of the boundary layer and beyond. Here, the z gradients are negligible. The pressure gradient is, as usual, known from the edge conditions. Thus, the difference form of Eq. (2) in this region of the boundary layer, where $j=J$ is

$$\frac{1}{h_{i+½}}U\left(\frac{u_{i+1,J}-u_{i,J}}{\Delta x}\right)=-\frac{1}{h_{i+½}}p_{x_{i+½}}$$
$$=\frac{1}{h_{i+½}}(u_eu_{ex})_{i+½}$$

Evaluating the edge velocity and its gradient at $(i+½)$

$$U(u_{i+1,J}-u_{i,J})=\left(\frac{u_{e_i}+u_{e_{i+1}}}{2}\right)(u_{e_{i+1}}-u_{e_i})$$

which gives, for the unknown velocity $u_{i+1,J}$

$$u_{i+1,J}=\left\{\frac{[(u_{e_i}+u_{e_{i+1}})/2]}{U}\right\}(u_{e_{i+1}}-u_{e_i})+u_{i,J} \tag{5}$$

where $u_{i,J}=u_{e_i}$ if we assume that we are advancing away from a station with a correct velocity profile. Thus, if the extrapolated value, U, is not equal to the correct midpoint value, $[(u_{e_i}+u_{e_{i+1}})/2]$, the computed value at the point J will be in error. The more severe the pressure gradient, i.e. the stronger the variation of u_e, the more the calculated value will deviate from the true value.

Calculations were performed for the boundary layer on a prolate spheroid of thickness ratio 1/4. The manifestation of the extrapolation error is evident in a lack of exponential decay of the u-velocity within the boundary layer to its edge value. With a value of the last z-station larger than the boundary-layer thickness, a calculation using the box method with Newton iteration reproduced the freestream velocity exactly. The Crank-Nicolson integration with extrapolation

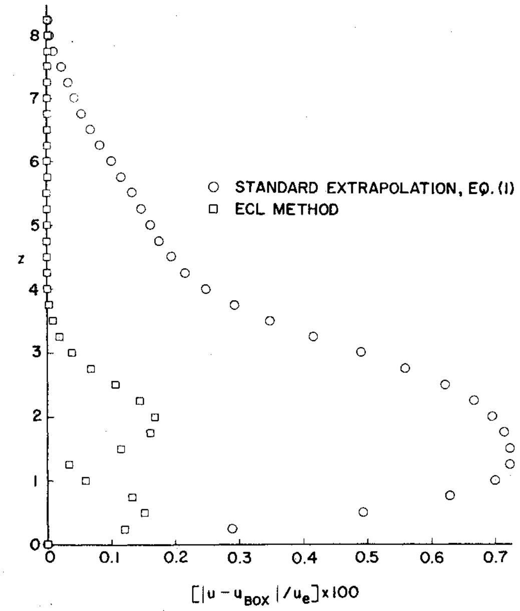

Fig. 2 Comparison of calculated values with box method results.

produced a velocity which gradually approached the freestream value, and only at the imposed boundary point was u exactly equal to u_e (see Fig. 2 and the discussion pertaining to it for details).

To emphasize this effect, a more drastic test was run where a jump in the edge values was imposed. The symmetry plane boundary layer on the prolate spheroid at 2° incidence was computed by the box method and used as the initial stations for an axisymmetric (i.e., 0° incidence) Crank-Nicolson calculation. The result was a velocity overshoot in the freestream region of the finite-difference grid. The maximum value of this overshoot was calculable directly from the formula given by Eq. (5).

This difficulty arises because Eq. (2), when solved with the linearized value, is inconsistent with the equation known to be valid in a certain region of the flow, i.e., the inviscid region. The simple solution which presents itself is to correct the linearized value so as to make the equation consistent for values of z at the boundary-layer edge and greater. Thus, the method is called edge-corrected linearization, ECL. Stated another way, this means that all the information available about the flow should be utilized, which is not the case when Eqs. (1) are used.

To implement this procedure, the linearized value, U, given by Eq. (1a) must be corrected to give the correct edge value at points $(i+\frac{1}{2},J)$, given by the average of the known edge values at (i) and $(i+1)$. A simple form that accomplishes this is

$$U_c = \frac{[(u_{e_i}+u_{e_{i+1}})/2]}{[(3/2)u_{e_i}-(1/2)u_{e_{i-2}}]}\,U = C\cdot U \tag{6}$$

This makes the edge value of U_c equal to the correct average value, and proportionally weights the extrapolation throughout the rest of the boundary layer. It is simple to accomplish computationally, and can be shown to not only retain the second-order accuracy of Eq. (1a), but to actually enhance it. Expanding both the numerator and denominator of C above, and combining terms yields:

$$U_c = u_{i+\frac{1}{2}} + h^2\left(\frac{2u_{i+\frac{1}{2}}u''_{e_{i+\frac{1}{2}}}}{u_{e_{i+\frac{1}{2}}}} - \frac{3}{2}u''_{i+\frac{1}{2}}\right) \tag{7}$$

At the edge, the truncation error is exactly correct for the average of the two known values [which is smaller in magnitude than that given by Eq. (1a) by a factor of 3], and should be smaller throughout the boundary layer, due to the opposite signs of the two terms in the truncation error given by Eq. (7).

This new ECL value of $u_{i+\frac{1}{2}}$ is to be used in Eq. (4) in place of the value of U appearing there. It remains to determine what the ECL value of $w_{i+\frac{1}{2}}$ should be. Unfortunately no reference value equivalent to u_e exists for determining the correct value of w at one point, since w is not determined from a two-point boundary-value problem. Consequently, the ECL value for $w_{i+\frac{1}{2}}$ was assumed to be

$$W_c = C^n \cdot W \tag{8}$$

and tested for values of $n=1/2$, 1,2. Very little change occurred in the results to be presented (less than one percent in skin friction), so the obvious relation of $n=1$ was chosen.

This procedure has been tested in the calculation of the boundary-layer flow past a prolate spheroid of thickness ratio 1/4, for which the potential solution is known exactly. As a means of comparison, the same problem was calculated using the box method.[1] The initial stations needed to start the extrapolating scheme were obtained from the box method solutions. Figure 1 shows a plot of the nondimensional skin friction at the wall τ_w, computed using the box and ECL methods. As can be seen, there is very little difference between the curves except approaching separation when the iteration of the box method is essential. Figure 2 shows the effect of not using the ECL method. The standard linearization, Eqs. (1), was used, and the difference between values of the velocity computed using the box method is plotted. The difference between ECL and box results is also shown. The differences are small in either case, but whereas the ECL values approach the edge values (at $z\simeq 4$) it is obvious that the behavior of the standard linearization is wrong. The effect of this incorrect outer behavior of the velocity is hardly felt on τ_w; it caused a maximum of two-percent change from the ECL value. However, since the ECL method reproduced the entire boundary-layer profile, the use of the new method was warranted.

The use of this linearization method should not be restricted to boundary-layer problems; it can be used in any parabolic marching problem where there is variation in the imposed boundary condition. It effectively accomplishes a normalization in the linearization of the equations rather than in the variables themselves. The results obtained show that calculations made with the ECL method reproduce, very closely, highly accurate solutions to the nonlinear equations being solved.

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